CLASSIFYING COMPLEMENTED SUBSPACES OF L_p WITH ALSPACH NORM

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This paper is dedicated to Professor Dale E. Alspach on the occasion of his 60th Birthday. Professor Alspach was born in April 30, 1950. He was one of first three Ph.D. students to finish their theses under the direction of William Johnson. Dr. Alspach's major contribution to Banach Space Theory are the first example of a non-expansive map on a weakly compact, closed convex subset of a Banach space without a fixed point and classification results for complemented subspaces of classical Banach spaces such as $C[0,1]$ and L_p, and translation invariant subspaces of $L_1(G)$ for G Abelian. He is currently the Department Head at Oklahoma State University.

Abstract. Understanding the complemented subspaces of L_p has been an interesting topic of research in Banach space theory since 1960. 1999, Alspach proposed a systematic approach to classifying the subspaces of L_p by introducing a norm given by partitions and weights. This paper shows that with Alspach Norm we are able to classify some complemented subspaces of $L_p, 2 < p < \infty$.

1. Introduction

Since 1960s understanding the complemented subspaces of L_p has been an interesting topic of research in Banach space theory. Early in the work only obvious combinations of ℓ_p and ℓ_2 were known to give examples. In 1972, Rosenthal's paper on sums of independent random variables was seminal. He created X_p and B_p spaces in this paper. In 1975, Schechtman proved that, up to isomorphism, there are infinitely many complemented subspaces of L_p by constructing tensor products of X_p spaces and in 1979, Bourgain, Rosenthal, and Schechtman proved that, up to isomorphism, there are uncountably many complemented subspaces of L_p. 1999, Alspach proposed a systematic approach to classifying the subspaces of L_p by introducing a norm given by partitions and weights. His proposal was the following:

Let A be a countable set and $P = \{N_i\}$ be a partition of A and $W : A \to (0,1]$ be a function, which we refer to as the *weights*. Let $x_j \in \mathbb{R}$ for all $j \in A$. Define

$$\|(x_j)_{j\in A}\|_{(P,W)} = \left(\sum_{N\in P}\left(\sum_{j\in N} x_j^2 w_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

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Suppose that $(P_k, W_k)_{k\in K}$ is a family of pairs of partitions and weights as above. Define a (possibly infinite) norm on the real-valued functions on A, $(x_i)_{i\in A}$, by

$$\|(x_i)\| = \sup_{k\in K} \|(x_i)\|_{(P_k,W_k)}$$

and let X be the subspace of elements of finite norm. In this case we say that X has an *Alspach Norm*.

It is important to note that a space with the Alspach norm has a natural unconditional basis and spaces with finite Alspach norm are Banach spaces [AT1]. Alspach norm provides sequence space realizations for some function spaces. Alspach's approach gives a unified description of many well known complemented subspaces of L_p. It is proved that the class of spaces with such norms is stable under $(p, 2)$ sums [AT1]. 2006, Alspach and Tong proved that subspaces of L_p, $p > 2$, with unconditional bases have equivalent partition and weight norms[AT2]. In this article we will explain what the conditions on partitions and weights will produce certain known complemented subspaces of L_p. Our work is far from complete due to the scope of this approach. Classifying all complemented subspaces of L_p with unconditional basis is a big challenge and with Alspach norm we made some progress.

From now on we will always assume that $p > 2$. In the rest of the paper, we use Rosenthal's X_p space and B_p space many times, [R]. Here are the definitions which can be found in Force's dissertation [F]: X_p can be realized as the closed linear span in L_p of a sequence $\{f_n\}$ of independent symmetric three-valued random variables such that the ratios $\|f_n\|_2 / \|f_n\|_p$ approaches zero slowly. Another realization of X_p is as the set of all sequences $\{x_n\}$ in ℓ_p for which the weighted ℓ_2 norm $(\sum |w_n x_n|^2)^{1/2}$ is finite and (w_n) is a fixed sequence that goes to zero slowly. The Banach space B_p is of the form $(Y_1 \oplus Y_2 \oplus \cdots)_{\ell_p}$, where each space Y_n is defined similar to X_p but is isomorphic to ℓ_2, and $\{Y_n\}_{n=1}^{\infty}$ is chosen so that $\sup_{n\in\mathbb{N}} d(Y_n, \ell_2) = \infty$, where $d(Y_n, \ell_2)$ is the Banach-Mazur distance between Y_n and ℓ_2.

A partition is called *discrete* if every subset has only one element. A partition is called *indiscrete* if the partition is the whole set. A partition which is not discrete and not indiscrete is called a *regular* partition. Partition P_1 is called a *refinement* of P_2 if every element in P_2 is a union of elements in P_1. From now on, we treat the discrete partition with constant weight of 1 as trivial and it will be included in the discussion but will not be counted towards the number of partitions. For instance we say that Rosenthal's X_p space [R] is an example of Banach space with Alspach norm given by one partition and weights. In the context of subspaces of L_p with unconditional basis there is always a lower ℓ_p estimate and the discrete partition ensures that the spaces we consider also have a lower ℓ_p estimate.

2. One Regular Partition

Definition 2.1. Let A and $\mathcal{P} = (P, W)$ be defined as above. A Banach space X is said to have Alspach norm with one partition and weights if

$$\|(x_j)\|_X = \max\left\{ \left(\sum x_j^p\right)^{\frac{1}{p}}, \left(\sum_{N\in P}\left(\sum_{j\in N} x_j^2 w_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \right\}$$

There is a complete classification of the spaces with Alspach norm and one regular partition.

Proposition 2.2. *Let $P = \{N_i : i \in B\}$ where B is an index set. Let $|N_i|$ be the cardinality of N_i. Let $I = \{i : |N_i| = \infty\}$. Then*

(1) *If $|I| < \infty$, then X is isomorphic to one of ℓ_p, X_p, ℓ_2, or $\ell_2 \oplus \ell_p$.*
(2) *If $|I| = \infty$, then X is isomorphic to one of ℓ_p, X_p, $\ell_2 \oplus \ell_p$, B_p, $(\sum \ell_2)_{\ell_p}$, $(\sum \ell_2)_{\ell_p} \oplus X_p$, $B_p \oplus X_p$, or $(\sum X_p)_{\ell_p}$.*

To prove the proposition we need following three lemmas. Below $X_i = X_i^{|N_i|} = [e_j : j \in N_i]$ where (e_j) is the natural basis of X.

Lemma 2.3. *Let I be defined as above. then*

$$X \sim \left(\sum_{i\in I} X_i\right)_{\ell_p} \oplus \left(\sum_{i\notin I} X_i^{|N_i|}\right)_{\ell_p}$$

Lemma 2.4. *If $B \setminus I$ is finite, then*

$$\ell_p \oplus \left(\sum_{i\notin I} X_i^{|N_i|}\right)_{\ell_p} \sim \ell_p$$

Lemma 2.5. *If $B \setminus I$ is infinite, then*

$$\left(\sum_{i\notin I} X_i^{|N_i|}\right)_{\ell_p} \sim \ell_p$$

Proof: For each $i \notin I$, X_i is a finite dimensional version of one of the spaces considered by Rosenthal and thus is isomorphic to a complemented subspace of ℓ_p and the norm of the projection is independent of i. This implies

$$\left(\sum X_i^{|N_i|}\right)_{\ell_p} \overset{c}{\hookrightarrow} \left(\sum \ell_p\right)_{\ell_p}.$$

Since $(\sum \ell_p)_{\ell_p} \sim \ell_p$, then

$$\left(\sum X_i^{|N_i|}\right)_{\ell_p} \overset{c}{\hookrightarrow} \ell_p.$$

Since every infinite dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p, then

$$\left(\sum X_i^{|N_i|}\right)_{\ell_p} \sim \ell_p.$$

After a messy computation based on splitting the argument into several cases depending on the isomorphic type of the ℓ_p sum of X_i for $i \in I$ the results in the proposition follow.

3. Admissible Partitions

Definition 3.1. A family of partitions and weights $\mathcal{P}$ is said to be *admissible* if there are partitions and weights (P_0, W_0), (P_1, W_1), and (P_2, W_2) in $\mathcal{P}$ such that (P_0, W_0) is the discrete partition with weight constantly 1, (P_1, W_1) is a regular partition and weight and P_2 is the indiscrete partition with weight $W_2 = (w_{2,j})$.

We have following result

Proposition 3.2. *Assume X be a sequence space of finite Alspach norm with an admissible family of partitions and weights and only one regular partition and weight. Then*

1. *If $\inf_j w_{2,j} \geq \delta > 0$, then $X \sim \ell_2$.*
2. *Suppose $\sum_j (w_{2,j})^{\frac{2p}{p-2}} < \infty$. Let $P_1 = \{N_i : i \in \mathbb{N}\}$. Let $|N_i|$ be the cardinality of N_i. Let $I = \{i : |N_i| = \infty\}$. Then X is isomorphic to one of the spaces listed in Proposition 2.2 (1).*
3. *If we combine first two cases, i.e. there is some $\delta > 0$, such that $\{j : w_{2,j} \geq \delta\}$ and $\{j : w_{2,j} < \delta\}$ are infinite and $\sum_{w_{2,j}<\delta} (w_{2,j})^{\frac{2p}{p-2}} < \infty$, then X is a direct sum of ℓ_2 from (1) and one of the spaces from (2).*

Proof:

1. Since $\inf w_{2j} \geq \delta > 0$, then
$$\delta \left(\sum_j x_j^2 \right)^{\frac{1}{2}} \leq \|(x_j)\|_X \leq \left(\sum_j x_j^2 \right)^{\frac{1}{2}}$$
2. Since $\sum_j (w_{2,j})^{\frac{2p}{p-2}} < \infty$, then we can apply Hölder's inequality
$$\left(\sum x_j^2 w_{2,j}^2 \right)^{\frac{1}{2}} \leq \left(\sum x_j^p \right)^{\frac{1}{p}} \left(\sum w_{2,j}^{\frac{2p}{p-2}} \right)^{\frac{p-2}{p}}$$
Thus
$$\|(x_j)\|_X \sim \|(x_j)\|_{P_1}$$
Apply the results from proposition 2.2 to the rest of the proof of (2).

4. Two Partitions Related by Refinement

Let A be any index set. Let P_1 and P_2 be two partitions of A and let $W_1 = (w_{1,k})$ and $W_2 = (w_{2,k})$ be two sequences of weights. Assume that P_1 is a refinement of P_2 and that $w_{1,j} \geq w_{2,j}$ for all $j \in A$. For a fixed $N \in P_1$, notice that

$$\begin{aligned} W_N &= \sup \left\{ \left(\sum x_j^2 w_{2,j}^2 \right)^{\frac{1}{2}} / \left(\sum x_j^2 w_{1,j}^2 \right)^{\frac{1}{2}} \right\} \\ &\leq \sup_{k \in N} \frac{w_{2,k}}{w_{1,k}} \sup \left\{ \left(\sum x_j^2 w_{1,j}^2 \right)^{\frac{1}{2}} / \left(\sum x_j^2 w_{1,j}^2 \right)^{\frac{1}{2}} \right\} \\ &= \sup_{k \in N} \frac{w_{2,k}}{w_{1,k}} \end{aligned}$$

the supremum in the first two lines are taken over all sequences (x_j) for $x_j = 0, j \notin N$ and $x_j \neq 0$ for finitely many j. Taking $x_j = 1$ and $x_k = 0$ for $k \neq j$, shows

that $W_N = \sup_{k\in N} \frac{w_{2,k}}{w_{1,k}}$.

Now notice that for fixed $M \in P_2$,

$$\sum_{N\subset M}\sum_{j\in N} x_j^2 w_{2,j}^2 \leq \sum_{N\subset M} W_N^2 \sum_{j\in N} x_j^2 w_{1,j}^2$$
$$\leq \left(\sum_{N\subset M} W_N^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}} \left(\sum_{N\subset M}\left(\sum_{j\in N} x_j^2 w_{1,j}^2\right)^{p/2}\right)^{2/p}$$

Then by Using Hölder's inequality

$$\|(x_j)\| = \max\left\{\left(\sum x_j^p\right)^{\frac{1}{p}}, \left(\sum_{N\in P_1}\left(\sum_{i\in N} x_i^2 w_{1i}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_{M\in P_2}\left(\sum_{j\in M} x_j^2 w_{2j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}$$
$$= \max\left\{\left(\sum x_j^p\right)^{\frac{1}{p}}, \left(\sum_{N\in P_1}\left(\sum_{i\in N} x_i^2 w_{1i}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_{M\in P_2}\left(\sum_{N\subset M}\sum_{j\in N} x_j^2 w_{2j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}$$
$$\leq \max\left\{\left(\sum x_j^p\right)^{\frac{1}{p}}, \left(\sum_{N\in P_1}\left(\sum_{i\in N} x_i^2 w_{1i}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_{M\in P_2}\left(\sum_{N\subset M} W_N^2 \sum_{j\in N} x_j^2 w_{1j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}$$
$$\leq \max\left\{\left(\sum x_j^p\right)^{\frac{1}{p}}, \left(\sum_{N\in P_1}\left(\sum_{i\in N} x_i^2 w_{1i}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_{M\in P_2}\left(\sum_{N\subset M} W_N^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2}}\left(\sum_{N\subset M}\sum_{j\in N} x_j^2 w_{1j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}$$

We have following result:

Proposition 4.1. *Let partition P_1 be a refinement of partition P_2. Let $W_1 = (w_{1,k})$ and $W_2 = (w_{2,k})$ be two corresponding sequences of weights. Assume that $w_{1,j} \geq w_{2,j}$ for all $j \in A$*

If

$$\sup_M \left\{\sum_{N\subset M} W_N^{\frac{2p}{p-2}}\right\} < \infty$$

then X can be classified by the behavior of W_1.

Proposition 4.2. *Let X be the Banach space with finite norms given by two partitions P_1 and P_2 and we also assume for every $N \in P_1$ there is an $M \in P_2$ such that $N \subset M$ and $w_{1k} \geq w_{2k}$ for all $k \in N$. We have following results:*

(1) *Assume* $\inf(w_{2k}) = \delta > 0$
 (a) *If* $|P_2| < \infty$ *we have*

$$\delta |P_2|^{\frac{1}{p}-\frac{1}{2}} \|x_j\|_2 \leq \delta \left(\sum_{M\in P_2}\left(\sum_{j\in M} x_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq \|(x_j)\|_X \leq \|(x_j)\|_2 .$$

Therefore,

$$X \sim \ell_2.$$

(b) *If* $|P_2| = \infty$ *and* $|M| < \infty$ *for all* $M \in P_2$,

$$\|x_j\|_X \sim \left(\sum_{M \in P_2} \left(\sum_{j \in M} x_j^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

Let Y_n *represent the space generated by the inner sum which implies that* $X \sim (\sum Y_n)_p$. *Since* X *is infinite dimensional we conclude*

$$X \sim \ell_p.$$

(c) *If* $|P_2| = \infty$, $|M| = \infty$ *for at least one and at most finitely many* $M \in P_2$ *Then* X *will be a direct sum of the spaces,*

$$X \sim \ell_2 \oplus \ell_p.$$

(d) *If* $|P_2| = \infty$ *and* $|M| = \infty$ *for infinitely many* $M \in P_2$, *then*

$$\|x_j\|_X \sim \left(\sum_{|M|=\infty} \left(\sum_{j \in M} x_j^2 \right)^{\frac{p}{2}} + \sum_{|M|<\infty} \left(\sum_{j \in M} x_j^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

and

$$X \sim \left(\sum \ell_2 \right)_{\ell_p}.$$

(2) *Assume* $\inf(w_{1k}) = \gamma > 0$ *and* $\inf(w_{2k}) = 0$

(a) *If* $|P_1| < \infty$, *we have*

$$\gamma |P_1|^{\frac{1}{p}-\frac{1}{2}} \|x_j\|_2 \le \gamma \left(\sum_{N \in P_1} \left(\sum_{j \in N} x_j^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \le \|(x_j)\|_X \le \|(x_j)\|_2$$

so

$$X \sim \ell_2.$$

(b) *If* $|P_1| = \infty$ *and* $|N| < \infty$ *for all* $N \in P_1$, *and for every* $M \in P_2$, *let* $C_M = \sum_{n \in M} w_{2n}^{\frac{2p}{p-2}} < \infty$ *and assume* $\sup_{M \in P_2} C_M < \infty$, *then*

$$X \sim \ell_p$$

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